

Invariant algebraic surfaces of the FitzHugh-Nagumo system

Liwei Zhang, Jiang Yu*

Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, China

E-mail: zhangliwei01@sjtu.edu.cn; jiangyu@sjtu.edu.cn;

Abstract: In this paper, we characterize all the irreducible Darboux polynomials and polynomial first integrals of FitzHugh-Nagumo (F-N) system. The method of the weight homogeneous polynomials and the characteristic curves is widely used to give a complete classification of Darboux polynomials of a system. However, this method does not work for F-N system. Here by considering the Darboux polynomials of an assistant system associated to F-N system, we classified the invariant algebraic surfaces of F-N system. Our results show that there is no invariant algebraic surface of F-N system in the biological parameters region.

Keywords: Darboux polynomial, integrability, FitzHugh-Nagumo system

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1 Introduction

In this paper, we consider the FitzHugh-Nagumo (F-N) system

$$u_t = u_{xx} - f(u) - v, v_t = \varepsilon(u - \gamma v), \quad (1.1)$$

where $f(u) = u(u-1)(u-a)$, and $0 < a < \frac{1}{2}$, $\varepsilon > 0$, $\gamma > 0$ are biological parameters. We say, u presents the voltage inside the axon at position $x \in \mathbb{R}$ and time t ; v presents a part of trans-membrance current that is passing slowly adapting iron channels.

These equations were introduced in papers of FitzHugh [16] and Nagumo et al. [9]. FitzHugh [16] simplified the 4-dimensional Hodgkin-Huxley (H-H) system into a planar system which is called Bonhoeffer-Van der Por system (BVP system for short). In [16], the author considered the excitable and oscillatory behavior of BVP system, and showed the underlying relationship between BVP system and H-H system. By the method that FitzHugh used in [16] and the Kirchhoff's law, Nagumo et al. [9] considered the propagation of the excitation along the nerve axon into H-H system, then H-H system becomes the partial differential system (1.1). From then on, FitzHugh-Nagumo (F-N) system (1.1) has been studied extensively in the literature, and it becomes one of

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the simplest models describing the excitation of neural membranes and the propagation of nerve impulses along an axon. It has been attracting lots of attentions about the existence, uniqueness and stability of this equation's travelling wave, such as in [1, 6–9, 17] and the references therein.

If we assume that the travelling wave of F-N system is a bounded solution $(u, v)(x, t)$ with $x, t \in \mathbb{R}$, satisfying $(u, v)(x, t) = (u, v)(x + ct)$, where c is a constant denoting the wave speed. Substituting $u = u(x + ct)$, $v = v(x + ct)$ into system (1.1), then we have the following ordinary differential system

$$\begin{aligned}\dot{x} &= z = P(x, y, z), \\ \dot{y} &= b(x - dy) = Q(x, y, z), \\ \dot{z} &= x(x - 1)(x - a) + y + cz = R(x, y, z),\end{aligned}\tag{1.2}$$

where the dot denotes derivative with respect to τ with $\tau = x + ct$, and $x = u$, $y = v$, $z = \dot{u}$, $b = \frac{\varepsilon}{c}$, $d = \gamma$.

In this paper, we focus on Darboux polynomials of 3-dimensional F-N system (1.2), with arbitrary parameters $a, b, c, d \in \mathbb{R}$. The Darboux polynomials and Darboux theory of integrability are considered as one of the important tools to look for first integral, and they also provide a relationship between the integrability of polynomial vector fields and the number of algebraic invariant surfaces of the system (see more details about the Darboux integrability in [5, 18]). There have been lots of papers discussing a system's invariant algebraic surfaces (Darboux polynomials), such as [2, 3, 13, 14, 19]. By studying the first integral or the invariant surface of a system, we can make a more precise analysis of the topological structure of dynamics of the system.

For F-N system (1.2), Jaume Llibre and Clàudia Valls [10] studied the system's analytic first integrals. When parameters $b = c = 0$, they obtained the analytic first integrals, which is the same as the Darboux polynomials with zero cofactor we have obtained in this paper. For F-N system (1.1), if we don't consider the propagation of the excitation along the nerve axon, i.e. $u_{xx} = b = \text{constant}$, then it becomes the following planar F-N system

$$\begin{aligned}\dot{x} &= x(1 - x)(x + a) - y + b, \\ \dot{y} &= d(x - cy),\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$ are parameters. In [11], the authors studied the Liouvillian integrability of the planar F-N system. Because the Liouvillian integrability is equivalent to the Darboux integrability for the planar polynomial vector fields, they classified the invariant algebraic curves of planar F-N system.

The method we used in this paper was introduced in [5]. This method contains the use of weight homogeneous polynomials and the method of characteristic curves for solving linear partial differential equations. And this method has been widely used to deal with the invariant algebraic surfaces of many famous systems, such as Lorenz system [12], generalized Lorenz system [15], Muthuswamy-chua system [14] et al.

According to the characteristic curve method which will be introduced in the next section, the critical step is to construct linear partial differential operator in order to obtain a pair of characteristic curves. However, it is difficult to find a linear differential operator which is valid for system (1.2). To overcome this difficulty, we construct an assistant system (3.1) associated to system (1.2). For system (3.1), we can find the linear differential operator to classify the Darboux polynomials by the characteristic curve method. Thus, based on the relationship between assistant system (3.1) and system (1.2), we give a complete classification of the invariant algebraic surfaces of F-N system (1.2).

In the following, we provide some definitions which is necessary to our proof.

Definition 1. A polynomial $f(x, y, z) \in \mathbb{C}[x, y, z]$ and $f(x, y, z) \not\equiv 0$, the ring of the complex coefficient polynomials in x, y, z is called a Darboux polynomial for the F-N system (1.2) if

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = kf$$

for some real polynomial $k(x, y, z)$, and $k(x, y, z)$ is called cofactor of f .

It is easy to know that the degree of $k(x, y, z)$ is smaller than the degree of system (1.2), where the degree of system (1.2) equals to max degree of $\{P, Q, R\}$. If $f(x, y, z)$ is a Darboux polynomial, then the set $\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$ is invariant under the flow of system (1.2).

Definition 2. If $f(x, y, z)$ is a Darboux polynomial, the set $\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$ is called an invariant algebraic surface.

We usually call simply $f(x, y, z) = 0$ an invariant algebraic surface.

Definition 3. A real function $H(x, y, z, t) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ is called an invariant of system (1.2), if $H(x, y, z, t) = \text{constant}$ on all solution curves $(x(t), y(t), z(t))^T$ of system (1.2). If the invariant H is independent of the time, then it is called a first integral.

If $H(x, y, z, t)$ is differentiable in \mathbb{R}^3 , then H is an invariant of system (1.2) if and only if

$$\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q + \frac{\partial H}{\partial z}R = 0.$$

Definition 4. A function $H(x, y, z)$ is called an algebraic function, if it satisfies the algebraic equation

$$f_0 + f_1C + f_2C^2 + \dots + f_{n-1}C^{n-1} + C^n = 0,$$

where $f_i(x, y, z)$, $(i = 0, 1, \dots, n-1)$ are given rational functions, and n is the smallest positive integer for which such a relation holds.

Obviously, any rational and polynomial functions are algebraic.

Definition 5. *If the first integral H is a polynomial (resp. a rational function or an algebraic function), then it is called a polynomial first integral (resp. a rational first integral or an algebraic first integral).*

Definition 6. *Let $H_1(x, y, z)$ and $H_2(x, y, z)$ be two first integrals of system (1.2). H_1 and H_2 are called independent, if their gradients are linearly independent vectors for all points $(x, y, z) \in \mathbb{R}^3$ except for a zero Lebesgue measure sets.*

The F-N system (1.2) is called polynomial integrable, or rational integrable or algebraic integrable if it has two independent polynomial first integrals, or rational first integrals or algebraic first integrals.

Definition 7. *Let \mathcal{S} be the set of all Darboux polynomials for system (1.2). A set \mathcal{T} is called a minimum subset of \mathcal{S} , if every element of \mathcal{S} is the finite product of the elements of \mathcal{T} , and the finite addition of the elements of \mathcal{T} with the same cofactor. Furthermore we call every element of \mathcal{T} a generator of \mathcal{S} .*

According to the Definition 7, the generators for the set of a system's Darboux polynomials are irreducible Darboux polynomials.

The following proposition indicts the reason why we find all the irreducible Darboux polynomials.

Proposition 1.1. *Assume that $f(x) \in \mathbb{C}[x]$ has an irreducible decomposition, saying $f(x) = f_1^{l_1} \cdots f_m^{l_m}$ with $f_i \in \mathbb{C}[x]$ and $l_i \in \mathbb{N}$ for $i \in \{1, \dots, m\}$. Then $f(x)$ is a Darboux polynomial of a polynomial vector field if and only if f_i for $i = 1, \dots, m$ are Darboux polynomials of the system. Moreover, if $k(x)$ and $k_i(x)$ are cofactors of $f(x)$ and $f_i(x)$ for $i = 1, \dots, m$ respectively, then $k(x) = l_1 k_1(x) + \cdots + l_m k_m(x)$.*

In the following, we state the main results in this paper.

Theorem 1. *The F-N system (1.2) has six generators for the set of Darboux polynomials listed in the Table 1:*

Corollary 1.1. (a) *The F-N system (1.2) has polynomial first integrals if and only if $b = c = 0$.*

(b) *The F-N system (1.2) is polynomial integrable.*

This paper is organised as follows. In Section 2, we introduce the method of characteristic curves for solving linear partial differential equation and the weight homogeneous polynomials; in Section 3, we present the proof of Theorem 1 and Corollary 1.1.

Remark 1. *Theorem 1 implies that there is no invariant algebraic surface of F-N system in the biological parameters region $0 < a < \frac{1}{2}$, $\varepsilon > 0$, $\gamma > 0$. Corollary 1.1 coincides with the results in [10].*

Darboux polynomials	Cofactors	Parameters
$\frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz + (\frac{1}{9}c^2 - 1)x^2$	$\frac{4}{3}c$	$a = -1, bd = -c, b = \frac{2}{27}c^3 - \frac{1}{3}c, c \neq 0$
$\frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz + (\frac{1}{9}c^2 - 1)x^2 - \frac{1}{2}dy^2$	$\frac{4}{3}c$	$a = -1, bd = -\frac{2}{3}c, b = \frac{2}{27}c^3 - \frac{1}{3}c, c \neq 0$
$\frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz - \frac{2}{3}(a+1)x^3$ $+ (\frac{1}{9}c^2 + a)x^2 - \frac{2}{9}c(a+1)z - \frac{2}{3}(a+1)y$ $- \frac{2}{27}c^2(a+1)x$	$\frac{4}{3}c$	$a \neq -1, bd = -c, c \neq 0,$ $b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c$ $2c^2 + 3a^2 - 12a + 3 = 0$
$\frac{1}{2}x^4 - z^2 - \frac{1}{2}dy^2 + 2xy + \frac{2}{3}cxz - \frac{2}{3}(a+1)x^3$ $+ (\frac{1}{9}c^2 + a)x^2 - \frac{2}{9}c(a+1)z - \frac{1}{3}(a+1)y$ $- \frac{2}{27}c^2(a+1)x$	$\frac{4}{3}c$	$a \neq -1, bd = -\frac{2}{3}c, c \neq 0,$ $b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c$ $2c^2 + a^2 - 7a + 1 = 0$
y	0	$b = c = 0$
$\frac{1}{4}x^4 - \frac{1}{2}z^2 - \frac{1}{3}(a+1)x^3 + xy + \frac{1}{2}ax^2$	0	$b = c = 0$

Table 1: The generators for the set of Darboux polynomials of system (1.2)

2 The characteristic curve method

The characteristic curves method for solving linear differential system see for instance [4, 12].

Consider the following first-order linear partial differential equation

$$a_1(x, y, z)A_x + a_2(x, y, z)A_y + a_3(x, y, z)A_z + a_0(x, y, z)A = f(x, y, z), \quad (2.1)$$

where $A = A(x, y, z)$, $a_i, i \in \{0, 1, 2, 3\}$ and f are C^1 maps.

Definition 8. A curve $(x(t), y(t), z(t))$ in the xyz space is called a characteristic curve for the equation (2.1), if the vector $(a_1(x_0, y_0, z_0), a_2(x_0, y_0, z_0), a_3(x_0, y_0, z_0))^T$ is tangent to the curve at each point (x_0, y_0, z_0) on the curve.

Based on the definition, a characteristic curve is a solution of the system

$$\frac{dx}{dt} = a_1(x, y, z), \quad \frac{dy}{dt} = a_2(x, y, z), \quad \frac{dz}{dt} = a_3(x, y, z).$$

Without loss of generality, assuming $a_1(x, y, z) \neq 0$ locally, then the above system is reduced to the following system

$$\frac{dy}{dx} = \frac{a_2(x, y, z)}{a_1(x, y, z)}, \quad \frac{dz}{dx} = \frac{a_3(x, y, z)}{a_1(x, y, z)}. \quad (2.2)$$

The ordinary differential equation (2.2) is known as the *characteristic equation* of (2.1).

Suppose $g(x, y, z) = c_1, h(x, y, z) = c_2$ is a solution of equation (2.2) in the implicit form, where c_1 and c_2 are arbitrary constants. Taking the change of the variables

$$u = x, \quad v = g(x, y, z), \quad w = h(x, y, z),$$

then we have

$$x = u, \quad y = p(u, v, w), \quad z = q(u, v, w).$$

Hence the linear partial differential equation (2.1) becomes the following ordinary differential equation in u for fixed v, w ,

$$\bar{a}_1(u, v, w)\bar{A}_u + \bar{a}_0(u, v, w)\bar{A} = \bar{f}(u, v, w), \quad (2.3)$$

where $\bar{a}_0, \bar{a}_1, \bar{A}$ and \bar{f} are a_0, a_1, A, f written in terms of u, v and w .

If $\bar{A} = \bar{A}(u, v, w)$ is a solution of (2.3), then by the variable transformation, we get

$$A(x, y, z) = \bar{A}(x, g(x, y, z), h(x, y, z)),$$

which is a solution of the linear partial differential equation (2.1). Moreover, we can obtain the general solution of (2.1) from the general solution of (2.3) by the variable transformation.

At last, we provide a definition of weight homogeneous polynomials, which is used in the proof of our results.

Definition 9. A polynomial $f(x_1, x_2, \dots, x_n)$ is said to be weight homogeneous if there exist $(s_1, s_2, \dots, s_n) \in \mathbb{N}$ and $d \in \mathbb{N}$ such that for all $\alpha \in \mathbb{R} \setminus \{0\}$,

$$f(\alpha^{s_1}x_1, \alpha^{s_2}x_2, \dots, \alpha^{s_n}x_n) = \alpha^d f(x_1, x_2, \dots, x_n),$$

where d is the weight degree of the polynomial, (s_1, s_2, \dots, s_n) is the weight exponents of the polynomial.

3 The proof of Theorem 1

In this section, we will present our proof of Theorem 1 on the basis of characteristic curve method. In order to obtain a pair of characteristic curves, we need to construct a linear partial differential operator. However, we find that it is impossible to get some operator for F-N system (1.2). Instead, we consider an assistant system corresponding to system (1.2) as follows

$$\begin{aligned} \dot{x} &= z, \\ \dot{y} &= b(x - dy) + mxz, \\ \dot{z} &= x(x - 1)(x - a) + y + cz. \end{aligned} \quad (3.1)$$

The difference between F-N system (1.2) and the assistant system (3.1) is whether there is a term mxz in $Q(x, y, z)$. By adding the term mxz , we can construct the linear partial differential operator L based on system (3.1),

$$L = z \frac{\partial}{\partial x} + mxz \frac{\partial}{\partial y} + x^3 \frac{\partial}{\partial z}. \quad (3.2)$$

We can check that its corresponding characteristic curves are weight polynomials of weight exponents $(1, 2, 2)$. And Darboux polynomial of system (3.1) can be expanded into a series of weight polynomials on the basis of the characteristic curves. Hence,

the characteristic curve method works for system (3.1). It is obvious that the Darboux polynomials of system (1.2) are the ones of system (3.1) with $m = 0$.

In the following, we use the method of characteristic curves to discuss the Darboux polynomials of the assistant system (3.1).

In order to obtain the linear partial differential operator, we make a weight change of variables, $(X, Y, Z, T) = (\alpha x, \alpha^2 y, \alpha^2 z, \alpha^{-1} t)$, $\alpha \in \mathbb{R} \setminus \{0\}$. The assistant system (3.1) becomes

$$\begin{aligned} X' &= Z, \\ Y' &= -\alpha b d Y + \alpha^2 b X + m X Z, \\ Z' &= X^3 - \alpha[(a+1)X^2 - Y - cZ] + \alpha^2 a X, \end{aligned} \quad (3.3)$$

where the prime denotes the derivative with respect to T .

The relationship between Darboux polynomials of system (3.1) and system (3.3) will be shown in the following.

Suppose that $f(x, y, z)$ is the Darboux polynomial of the assistant system (3.1) with cofactor $k(x, y, z)$. It is easy to prove that the degree of k is less than or equal to 2. Therefore, we can assume that the cofactor is in the form

$$k(x, y, z) = k_{200}x^2 + k_{110}xy + k_{011}yz + k_{101}xz + k_{020}y^2 + k_{002}z^2 + k_{100}x + k_{010}y + k_{001}z + k_0.$$

Set

$$\begin{aligned} F(X, Y, Z) &= \alpha^l f(\alpha^{-1}X, \alpha^{-2}Y, \alpha^{-2}Z), \\ K(X, Y, Z) &= \alpha^h k(\alpha^{-1}X, \alpha^{-2}Y, \alpha^{-2}Z). \end{aligned}$$

Expanding $f(x, y, z)$ and $k(x, y, z)$ by the order of weight degree with weight exponents $(1, 2, 2)$, where l, h are the highest weight degree in the expansion of $f(x, y, z)$ and $k(x, y, z)$, respectively. It is easy to check that $h = 4$. Then $F(X, Y, Z)$ is the Darboux polynomial of system (3.3) with cofactor $\alpha^{-3}K(X, Y, Z)$, since

$$\begin{aligned} \frac{dF}{dT}|_{(3.3)} &= \alpha^{l+1}k(\alpha^{-1}x, \alpha^{-2}y, \alpha^{-2}z)f(\alpha^{-1}x, \alpha^{-2}y, \alpha^{-2}z) \\ &= \alpha^{-3}K(X, Y, Z)F(X, Y, Z). \end{aligned}$$

In the view of $f = F|_{\alpha=1}$, the Darboux polynomials of F-N system (1.2) satisfy $f|_{m=0} = F|_{\alpha=1, m=0}$.

We remark that an arbitrary polynomial can be expanded into a series of weight homogeneous polynomials by the order of weight degree with the same weight exponents. Then, we can expand

$$F(X, Y, Z) = F_0(X, Y, Z) + \alpha F_1(X, Y, Z) + \alpha^2 F_2(X, Y, Z) + \cdots + \alpha^l F_l(X, Y, Z), \quad (3.4)$$

where F_j is a weight homogeneous polynomial with weight exponents $(1, 2, 2)$, $j = 0, 1, \dots, l$, and weight degree is $l - j$. Noticing that F is a Darboux polynomial of system (3.3) with the cofactor $\alpha^{-3}K$, we have from Definition 1 of Darboux polynomial

that

$$\begin{aligned}
& z \sum_{j=0}^l \alpha^j \frac{\partial F_j}{\partial x} + (mxz - \alpha bdy + \alpha^2 bx) \sum_{j=0}^l \alpha^j \frac{\partial F_j}{\partial y} \\
& + [x^3 - \alpha[(a+1)x^2 - y - cz] + \alpha^2 ax] \sum_{j=0}^l \alpha^j \frac{\partial F_j}{\partial z} \\
& = [\alpha^{-3}(k_{020}y^2 + k_{002}z^2 + k_{011}yz) + \alpha^{-2}(k_{110}xy + k_{101}xz) \\
& + \alpha^{-1}(k_{200}x^2 + k_{010}y + k_{001}z) + k_1x + \alpha k_0] \sum_{j=0}^l \alpha^j F_j.
\end{aligned}$$

For convenience, we use x, y, z and k_1 instead of X, Y, Z and k_{100} here.

Comparing the terms with $\alpha^{-3}, \alpha^{-2}, \alpha^{-1}$, we have $K(x, y, z) = \alpha^3 k_1 x + \alpha^4 k_0$, that is, the cofactor in (3.1) is $k(x, y, z) = k_1 x + k_0$.

Equating the terms with α^j ($j = 0, 1, \dots, l$), we get the following first linear partial differential equations

$$\begin{aligned}
L[F_0] &= k_1 x F_0, \\
L[F_1] &= k_1 x F_1 + k_0 F_0 + bdy \frac{\partial F_0}{\partial y} + [(a+1)x^2 - y - cz] \frac{\partial F_0}{\partial z}, \\
L[F_j] &= k_1 x F_j + k_0 F_{j-1} + bdy \frac{\partial F_{j-1}}{\partial y} + [(a+1)x^2 - y - cz] \frac{\partial F_{j-1}}{\partial z} \\
&\quad - bx \frac{\partial F_{j-2}}{\partial y} - ax \frac{\partial F_{j-2}}{\partial z}, \quad j = 2, 3, \dots, l+3,
\end{aligned} \tag{3.5}$$

when $j > l$, $F_j = 0$, and L is the linear partial differential operator of the form (3.2).

For equations in (3.5), the characteristic equations are

$$\frac{dy}{dx} = mx, \quad \frac{dz}{dx} = \frac{x^3}{z}.$$

The solutions of characteristic equations are

$$y - \frac{1}{2}mx^2 = c_1, \quad \frac{1}{4}x^4 - \frac{1}{2}z^2 = c_2, \tag{3.6}$$

where c_1 and c_2 are constants of integration.

According to the method of characteristics curve, we make the change of the variables

$$u = x, \quad v = y - \frac{1}{2}mx^2, \quad w = \frac{1}{4}x^4 - \frac{1}{2}z^2. \tag{3.7}$$

Then, we have

$$x = u, \quad y = v + \frac{1}{2}mu^2, \quad z = \pm \sqrt{\frac{1}{2}u^4 - 2w}.$$

In the following, without loss of generality we take

$$x = u, \quad y = v + \frac{1}{2}mu^2, \quad z = \sqrt{\frac{1}{2}u^4 - 2w}. \tag{3.8}$$

Through the change of variables (3.7) and (3.8), it follows from the first equation of (3.5) for fixed v and w that

$$\sqrt{\frac{1}{2}u^4 - 2w} \frac{d\overline{F}_0}{du} = k_1 u \overline{F}_0, \quad (3.9)$$

where \overline{F}_0 is the function F_0 written in the variables u, v and w . In the following, \overline{F}_j denote F_j written in u, v and w . Solving Equation (3.9), we have

$$\overline{F}_0(u, v, w) = \overline{G}_0(v, w) (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1},$$

where $\overline{G}_0(v, w)$ is an arbitrary smooth function. Then

$$F_0(x, y, z) = \overline{F}_0(u, v, w) = \overline{F}_0(x, y - \frac{1}{2}mx^2, \frac{1}{4}x^4 - \frac{1}{2}z^2),$$

is the solution of the first equation in (3.5). Similarly,

$$F_j(x, y, z) = \overline{F}_j(x, y - \frac{1}{2}mx^2, \frac{1}{4}x^4 - \frac{1}{2}z^2), \quad j = 1, \dots, l,$$

are the solutions of the other equations in (3.5).

Hence we write F_0 in variables x, y and z as follows

$$F_0(x, y, z) = \overline{G}_0(y - \frac{1}{2}mx^2, \frac{1}{4}x^4 - \frac{1}{2}z^2) (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1}. \quad (3.10)$$

From the transformation (3.7), we know that $v(x, y, z)$ and $w(x, y, z)$ are weight homogeneous polynomials of degree 2 and 4 with weight exponents $(1, 2, 2)$. Then we know from (3.10) that G_0 should be a weight polynomial of the following two forms for some $n \in \mathbb{N}$

$$G_0(x, y, z) = \overline{G}_0(v, w) = \sum_{i=1}^n a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i}, \quad (3.11)$$

or

$$G_0(x, y, z) = \overline{G}_0(v, w) = \sum_{i=0}^n a_i (y - \frac{1}{2}mx^2)^{2i} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i}. \quad (3.12)$$

In the following, we divide our proof into three parts. In section 3.1, we prove that the cofactor of Darboux polynomials of system (3.1) must be a constant, Then in section 3.2 we classify the Darboux polynomial when the cofactor is a nonzero constant. In section 3.3, we continue to discuss the Darboux polynomials with zero cofactor. In each section, we shall consider the two different forms of G_0 respectively.

3.1 The nonzero cofactor of Darboux polynomials

In this subsection, we shall prove the following lemma.

Lemma 3.1. *If $f(x, y, z)$ is a Darboux polynomial with nonzero cofactor of (3.1), then the cofactor is a constant, that is, $k(x, y, z) = k_0$ and $k_1 = 0$.*

Proof. First, take F_0 with G_0 in (3.11), i.e.

$$F_0 = \sum_{i=1}^n a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1}.$$

Substituting F_0 into the second equation of equation of (3.5), we obtain that

$$\begin{aligned} L[F_1] &= k_1 x F_1 \\ &+ \sum_{i=1}^n k_0 a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1} \\ &+ \sum_{i=1}^n b d (2i-1) a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1} \\ &+ \sum_{i=1}^n \frac{1}{2} m b d (2i-1) a_i (y - \frac{1}{2}mx^2)^{2i-2} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1} x^2 \\ &+ \sum_{i=1}^n \sqrt{2} (a+1) k_1 a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1-1} x^2 \\ &- \sum_{i=1}^n (a+1) (n-i) a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1} x^2 z \\ &- \sum_{i=1}^n \sqrt{2} k_1 a_i (y - \frac{1}{2}mx^2)^{2i} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1-1} \\ &+ \sum_{i=1}^n (n-i) a_i (y - \frac{1}{2}mx^2)^{2i} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1} z \\ &+ \sum_{i=1}^n \frac{1}{2} m (n-i) a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1} x^2 z \\ &- \sum_{i=1}^n \frac{\sqrt{2}}{2} m k_1 a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1-1} x^2 \\ &- \sum_{i=1}^n \sqrt{2} k_1 c a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1-1} z \\ &+ \sum_{i=1}^n c (n-i) a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1} z^2. \end{aligned}$$

By the transformations (3.7) and (3.8), we obtain the linear ordinary differential equation of \bar{F}_1 with respect to u for fixed v and w

$$\begin{aligned} \frac{d\bar{F}_1}{du} &= k_1 \bar{F}_1 \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} \\ &+ \sum_{i=1}^n k_0 a_i v^{2i-1} w^{n-i} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1} \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^n bd(2i-1)a_i v^{2i-1} w^{n-i} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1} \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} \\
& + \sum_{i=1}^n \frac{1}{2}mbd(2i-1)a_i v^{2i-2} w^{n-i} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1} \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} \\
& - \sum_{i=1}^n \sqrt{2}k_1 a_i v^{2i} w^{n-i} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1-1} \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} \\
& + \sum_{i=1}^n \sqrt{2}(a+1)k_1 a_i v^{2i-1} w^{n-i} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1-1} \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} \\
& - \sum_{i=1}^n \frac{\sqrt{2}}{2}mk_1 a_i v^{2i-1} w^{n-i} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1-1} \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} \\
& + \sum_{i=1}^n \frac{1}{2}m(n-i)a_i v^{2i-1} w^{n-i-1} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1} u^2 \\
& - \sum_{i=1}^n (a+1)(n-i)a_i v^{2i-1} w^{n-i-1} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1} u^2 \\
& + \sum_{i=1}^n (n-i)a_i v^{2i} w^{n-i-1} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1} \\
& - \sum_{i=1}^n \sqrt{2}k_1 c a_i v^{2i-1} w^{n-i} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1-1} \\
& + \sum_{i=1}^n (n-i)c a_i v^{2i-1} w^{n-i-1} (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1} \sqrt{\frac{1}{2}u^4 - 2w}.
\end{aligned}$$

Integrating this equation, we have

$$\begin{aligned}
\overline{F}_1 &= (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{\frac{\sqrt{2}}{2}k_1} (C \\
& + \sum_{i=1}^n k_0 a_i v^{2i-1} w^{n-i} \int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\
& + \sum_{i=1}^n bd(2i-1)a_i v^{2i-1} w^{n-i} \int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\
& + \sum_{i=1}^n \frac{1}{2}mbd(2i-1)a_i v^{2i-2} w^{n-i} \int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\
& - \sum_{i=1}^n \sqrt{2}k_1 a_i v^{2i} w^{n-i} \int (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{-1} \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^n (\sqrt{2}(a+1) - \frac{\sqrt{2}}{2}m) k_1 a_i v^{2i-1} w^{n-i} \int (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{-1} \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\
& - \sum_{i=1}^n (\frac{1}{3}(a+1) - \frac{1}{6}m)(n-i) a_i v^{2i-1} w^{n-i-1} u^3 \\
& + \sum_{i=1}^n (n-i) a_i v^{2i} w^{n-i-1} u \\
& - \sum_{i=1}^n \sqrt{2} k_1 c a_i v^{2i-1} w^{n-i} \int (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{-1} du \\
& + \sum_{i=1}^n (n-i) c a_i v^{2i-1} w^{n-i-1} \int \sqrt{\frac{1}{2}u^4 - 2w} du.
\end{aligned}$$

From the formulas in Appendix, every integrals in the expression of $\overline{F_1}$ can be expressed by the combination of the two integrals $\int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ and $\int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du$. Since F_1 is a weight homogeneous polynomial, $\int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ and $\int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ are not polynomials, then the coefficients of $\int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ and $\int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ should be vanish. Using the formulas in Appendix, we can write the term with the integral $\int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ in $\overline{F_1}$ as

$$\sum_{i=1}^n \frac{1}{4} k_1 a_i v^{2i} w^{n-i-1} \int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du = 0,$$

which implies $k_1 = 0$.

Secondly, taking F_0 with G_0 in (3.12), we have

$$F_0 = \sum_{i=1}^n a_i (y - \frac{1}{2}mx^2)^{2i} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} (\sqrt{2}x^2 + 2z)^{\frac{\sqrt{2}}{2}k_1}.$$

By the same way which we used in the first part, we can also prove $k_1 = 0$. \square

3.2 Darboux polynomials with nonzero cofactor $k_1 = 0, k_0 \neq 0$

In this subsection, we shall present the Darboux polynomials of system (1.2) with nonzero cofactor and the conditions for their existence.

Now by Lemma 3.1 and (3.10), we have $F_0 = G_0$, i.e.

$$F_0 = \sum_{i=1}^n a_i (y - \frac{1}{2}mx^2)^{2i-1} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i}, \tag{3.13}$$

or

$$F_0 = \sum_{i=0}^n a_i (y - \frac{1}{2}mx^2)^{2i} (\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i}. \tag{3.14}$$

Notice that the weight degree l of F_0 in the form (3.13) and (3.14) is $4n - 2$ and $4n$, respectively.

Remark 2. We claim that not all of a_i are zero, that is, $F_0 \not\equiv 0$. Otherwise, in view of $k_1 = 0$, it follows from (3.5) that $F_0 \equiv 0$ implies $F \equiv 0$.

Lemma 3.2. When F_0 is in the form (3.13), system (3.1) doesn't have Darboux polynomial with nonzero cofactor.

Proof. Substituting F_0 of (3.13) into the second equation of (3.5), we obtain that

$$\begin{aligned} L[F_1] = & \sum_{i=1}^n (k_0 + (2i-1)bd)a_i(y - \frac{1}{2}mx^2)^{2i-1}(\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i} \\ & + \sum_{i=1}^n (\frac{1}{2}m - (a+1))(n-i)a_i(y - \frac{1}{2}mx^2)^{2i-1}(\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1}x^2z \\ & + \sum_{i=1}^n (n-i)a_i(y - \frac{1}{2}mx^2)^{2i}(\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1}z \\ & + \sum_{i=1}^n c(n-i)a_i(y - \frac{1}{2}mx^2)^{2i-1}(\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1}z^2 \\ & + \sum_{i=1}^n \frac{1}{2}mbd(2i-1)a_i(y - \frac{1}{2}mx^2)^{2i-1}(\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i}x^2 \end{aligned}$$

Through the variable changes (3.7) and (3.8), we obtain the following ordinary linear differential equation with respect to u for fixed v and w

$$\begin{aligned} \frac{d\bar{F}_1}{du} = & \sum_{i=1}^n (k_0 + (2i-1)bd)a_i v^{2i-1} w^{n-i} \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} \\ & + \sum_{i=1}^n (\frac{1}{2}m - (a+1))(n-i)a_i v^{2i-1} w^{n-i-1} u^2 \\ & + \sum_{i=1}^n (n-i)a_i v^{2i} w^{n-i-1} \\ & + \sum_{i=1}^n c(n-i)a_i v^{2i-1} w^{n-i-1} \sqrt{\frac{1}{2}u^4 - 2w} \\ & + \sum_{i=1}^n \frac{1}{2}mbd(2i-1)a_i v^{2i-1} w^{n-i} \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}}. \end{aligned}$$

Using formulas presented in the Appendix, we solve this equation and obtain

$$\bar{F}_1 = \sum_{i=1}^n (k_0 + (2i-1)bd - \frac{4}{3}c(n-i))a_i v^{2i-1} w^{n-i} \int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du$$

$$\begin{aligned}
& + \sum_{i=1}^n \frac{1}{2} m b d (2i-1) a_i v^{2i-1} w^{n-i} \int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\
& - \sum_{i=1}^n \frac{1}{3} \left(\frac{1}{2} m - (a+1) \right) (n-i) a_i v^{2i-1} w^{n-i-1} u^3 \\
& + \sum_{i=1}^n (n-i) a_i v^{2i} w^{n-i-1} u \\
& + \sum_{i=1}^n \frac{1}{3} c (n-i) a_i v^{2i-1} w^{n-i-1} u \sqrt{\frac{1}{2}u^4 - 2w} + \overline{G}_1(v, w),
\end{aligned}$$

where $\overline{G}_1(v, w)$ is an arbitrary polynomial in v and w . $G_1(x, y, z) = \overline{G}_1(v, w)$ is a weight polynomial with weight degree even. Noticing that F_1 is a weight homogeneous polynomial of weight degree $4n-3$, then we have $\overline{G}_1(v, w) = 0$, and the coefficients of $\int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ and $\int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ must vanish. Then we have for $i = 1, 2, \dots, n$,

$$(k_0 + (2i-1)bd - \frac{4}{3}c(n-i))a_i = 0 \quad m b d a_i = 0, i = 1, 2, \dots, n, \quad (3.15)$$

and

$$\begin{aligned}
F_1 &= \sum_{i=1}^n \frac{1}{3} \left(\frac{1}{2} m - (a+1) \right) (n-i) a_i \left(y - \frac{1}{2} m x^2 \right)^{2i-1} \left(\frac{1}{4} x^4 - \frac{1}{2} z^2 \right)^{n-i-1} x^3 \\
&+ \sum_{i=1}^n (n-i) a_i \left(y - \frac{1}{2} m x^2 \right)^{2i} \left(\frac{1}{4} x^4 - \frac{1}{2} z^2 \right)^{n-i-1} x \\
&+ \sum_{i=1}^n \frac{1}{3} c (n-i) a_i \left(y - \frac{1}{2} m x^2 \right)^{2i-1} \left(\frac{1}{4} x^4 - \frac{1}{2} z^2 \right)^{n-i-1} x z.
\end{aligned}$$

Furthermore from Remark 2, $F_0 \neq 0$ implies that there is at least $a_i \neq 0$, then

$$k_0 = \frac{4}{3}c(n-i) - (2i-1)bd.$$

Consider F_2 in (3.5) with $j = 2$. We observe that there is only a term $k_0 F_1$ in (3.5) with $j = 2$, which includes k_0 . Hence, we always have $k_0 a_i$ after substituting F_1 into the equation. Without loss of generality, we can substitute $k_0 = \frac{4}{3}c(n-i) - (2i-1)bd$, $i = 1, 2, \dots, n$ into the equation. Then solving this ordinary differential equation with respect to u for fixed v and w , we get

$$\begin{aligned}
\overline{F}_2 &= \sum_{i=1}^n \left(\frac{4}{3}(n-i)c + bd - \frac{1}{3}c \right) (n-i) a_i v^{2i} w^{n-i-1} \int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\
&- \sum_{i=1}^n b(2i-1) a_i v^{2i-2} w^{n-i} \int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du
\end{aligned}$$

$$\begin{aligned}
& - \sum_{i=1}^n \frac{4}{3} c(n-i)(n-i-1) a_i v^{2i} w^{n-i-1} \int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\
& + \sum_{i=1}^n \frac{1}{2} \left(\frac{4}{9} (n-i)c^2 - \frac{1}{3} c^2 + a \right) (n-i) a_i v^{2i-1} w^{n-i-1} u^2 \\
& + \sum_{i=1}^n \frac{1}{2} (n-i)(n-i-1) a_i v^{2i+1} w^{n-i-2} u^2 \\
& - \sum_{i=1}^n \frac{1}{3} (a+1)(n-i)(n-i-1) a_i v^{2i} w^{n-i-2} u^4 \\
& + \sum_{i=1}^n \frac{1}{6} m(n-i)(n-i-1) a_i v^{2i} w^{n-i-2} u^4 \\
& + \sum_{i=1}^n \left(\frac{1}{18} (a+1)^2 - \frac{1}{18} m(a+1) + \frac{1}{72} m^2 \right) (n-i)(n-i-1) a_i v^{2i-1} w^{n-i-2} u^6 \\
& + \sum_{i=1}^n \left(\frac{4}{9} (n-i) + \frac{1}{3} \right) c(a+1)(n-i) a_i v^{2i-1} w^{n-i-1} \sqrt{\frac{1}{2}u^4 - 2w} \\
& + \sum_{i=1}^n \left(\frac{2}{9} (n-i) - \frac{1}{6} \right) m c(n-i) a_i v^{2i-1} w^{n-i-1} \sqrt{\frac{1}{2}u^4 - 2w} \\
& + \sum_{i=1}^n \frac{1}{3} c(n-i)(n-i-1) a_i v^{2i} w^{n-i-2} u^2 \sqrt{\frac{1}{2}u^4 - 2w} \\
& - \sum_{i=1}^n \left(\frac{1}{18} m - \frac{1}{9} (a+1) \right) c(n-i)(n-i-1) a_i v^{2i-1} w^{n-i-2} (u^4 - 4w) \sqrt{\frac{1}{2}u^4 - 2w} \\
& + \sum_{i=1}^n \frac{1}{3} c^2(n-i)(n-i-1) a_i v^{2i-1} w^{n-i-2} \left(\frac{1}{12} u^6 - w u^2 \right) + \overline{G}_2(v, w),
\end{aligned}$$

where $\overline{G}_2(v, w)$ is a polynomial.

Because F_2 is weight homogeneous polynomial of weight degree $4n - 4$, the coefficients of $\int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ vanish. Collecting the coefficients of $\int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ in the expression of \overline{F}_2 , we have

$$\sum_{i=1}^{n-1} ((bd + c)(n-i)a_i - b(2i+1)a_{i+1}) v^{2i} w^{n-i-1} - ba_1 w^{n-1} = 0, \quad (3.16)$$

which implies $ba_1 = 0$ and

$$(bd + c)(n-i)a_i - b(2i+1)a_{i+1} = 0, \quad i = 1, \dots, n-1.$$

If $b \neq 0$, then $a_1 = 0$, it inducts $a_i = 0, i = 1, \dots, n$. It contradicts with Remark 2.

If $b = 0$, (3.16) becomes

$$\sum_{i=1}^{n-1} c(n-i)a_i v^{2i} w^{n-i-1} = 0.$$

Then $c = 0$, it contradicts with (3.15) with $k_0 \neq 0$. \square

From Lemma 3.2, we shall take F_0 in (3.14) for Darboux polynomials with nonzero cofactor.

Remark 3. It follows from Lemma 3.2 that the highest weight degree in the expansion (3.4) of the Darboux polynomial f with nonzero cofactor should be $4n$, $n \in \mathbb{N}$.

Lemma 3.3. If F_0 is in the form (3.14), parameters in system (1.2) having Darboux polynomials with nonzero cofactors satisfy one of the following conditions:

- (1) $a = -1, bd = -c, b = \frac{2}{27}c^3 - \frac{1}{3}c, c \neq 0$;
- (2) $-\frac{1}{81}c^2 - \frac{1}{27}a^2 + \frac{4}{27}a - \frac{1}{27} = 0, bd = -c, b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c, c \neq 0$;
- (3) $a = -1, bd = -\frac{2}{3}c, b = \frac{2}{27}c^3 - \frac{1}{3}c, c \neq 0$;
- (4) $-\frac{1}{81}c^2 - \frac{1}{27}a^2 + \frac{4}{27}a - \frac{1}{27} = 0, bd = -\frac{2}{3}c, b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c, c \neq 0$.

Proof. Substituting F_0 of (3.14) into the second equation of (3.5), then solving it, we get the following equations by using formulas in the Appendix

$$\begin{aligned} \overline{F}_1 = & \sum_{i=0}^n (k_0 + 2ibd - \frac{4}{3}(n-i)c)a_i v^{2i} w^{n-i} \int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\ & + \sum_{i=0}^n mbdia_i v^{2i-1} w^{n-i} \int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\ & - \sum_{i=0}^n \frac{1}{3}(a+1)(n-i)a_i v^{2i} w^{n-i-1} u^3 \\ & + \sum_{i=0}^n (n-i)a_i v^{2i+1} w^{n-i-1} u \\ & + \sum_{i=0}^n \frac{1}{3}c(n-i)a_i v^{2i} w^{n-i-1} u \sqrt{\frac{1}{2}u^4 - 2w} \\ & + \sum_{i=0}^n \frac{1}{6}m(n-i)a_i v^{2i} w^{n-i-1} u^3 + \overline{G}_1(v, w), \end{aligned} \tag{3.17}$$

where $\overline{G}_1(v, w)$ is an arbitrary polynomial in v and w . $G_1(x, y, z) = \overline{G}_1(v, w)$ is a weight polynomial of weight degree even. However, F_1 is a weight homogeneous polynomial

of weight degree $4n - 1$, then $\overline{G}_1(v, w) = 0$, and the coefficients of $\int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ and $\int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ must vanish. Then we have $i = 0, 1, \dots, n$

$$(k_0 + 2ibd - \frac{4}{3}c(n - i))a_i = 0, \quad imbda_i = 0, \quad (3.18)$$

and from (3.17)

$$\begin{aligned} F_1 = & - \sum_{i=0}^n \frac{1}{3}(a+1)(n-i)a_i(y - \frac{1}{2}mx^2)^{2i}(\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1}x^3 \\ & + \sum_{i=0}^n (n-i)a_i(y - \frac{1}{2}mx^2)^{2i+1}(\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1}x \\ & + \sum_{i=0}^n \frac{1}{3}c(n-i)a_i(y - \frac{1}{2}mx^2)^{2i}(\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1}xz \\ & + \sum_{i=0}^n \frac{1}{6}m(n-i)a_i(y - \frac{1}{2}mx^2)^{2i}(\frac{1}{4}x^4 - \frac{1}{2}z^2)^{n-i-1}x^3. \end{aligned} \quad (3.19)$$

Due to Remark 2 and the first equation in (3.18), then

$$k_0 = \frac{4}{3}(n-i)c - 2ibd = \frac{4}{3}nc - 2i(bd + \frac{2}{3}c). \quad (3.20)$$

By the same method which has been used in the proof of Lemma 3.2, solving \overline{F}_2 in (3.5) with $j = 2$, we get

$$\begin{aligned} \overline{F}_2 = & \sum_{i=0}^n (\frac{4}{3}(n-i)c + bd - \frac{1}{3}c)(n-i)a_i v^{2i+1} w^{n-i-1} \int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\ & - \sum_{i=0}^n 2bia_i v^{2i-1} w^{n-i} \int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\ & - \sum_{i=0}^n \frac{4}{3}c(n-i)(n-i-1)a_i v^{2i+1} w^{n-i-1} \int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\ & + \sum_{i=0}^n \frac{1}{2}(\frac{1}{3}k_0c + \frac{2}{3}bdic - \frac{1}{3}c^2 + a)(n-i)a_i v^{2i} w^{n-i-1} u^2 \\ & + \sum_{i=0}^n \frac{1}{2}(n-i)(n-i-1)a_i v^{2i+2} w^{n-i-2} u^2 \\ & - \sum_{i=0}^n (\frac{1}{3}(a+1) + \frac{1}{8}m)(n-i)(n-i-1)a_i v^{2i+1} w^{n-i-2} u^4 \\ & + \sum_{i=0}^n (\frac{1}{18}(a+1)^2 - \frac{1}{36}m(a+1) + \frac{1}{72}m^2)(n-i)(n-i-1)a_i v^{2i} w^{n-i-2} u^6 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^n \left(\frac{1}{3} \left(-\frac{4}{3}(n-i) + 1 \right) (a+1) - \frac{1}{6}m \right) c(n-i) a_i v^{2i} w^{n-i-1} \sqrt{\frac{1}{2}u^4 - 2w} \\
& - \sum_{i=0}^n \frac{1}{9} c(a+1)(n-i)(n-i-1) a_i v^{2i} w^{n-i-2} (u^4 - 4w) \sqrt{\frac{1}{2}u^4 - 2w} \\
& + \sum_{i=0}^n \frac{1}{3} c(n-i)(n-i-1) a_i v^{2i+1} w^{n-i-2} u^2 \sqrt{\frac{1}{2}u^4 - 2w} \\
& + \sum_{i=0}^n \frac{1}{3} c^2(n-i)(n-i-1) a_i v^{2i} w^{n-i-2} \left(\frac{1}{12}u^6 - wu^2 \right) + \overline{G}_2(v, w),
\end{aligned}$$

where $\overline{G}_2(v, w)$ is a weight polynomial. Since F_2 is a weight homogeneous polynomial of weight degree $4n - 2$, the coefficients of $\int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ must vanish. Adjusting the coefficients of $\int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ in the expression of \overline{F}_2 , we have

$$\sum_{i=0}^n [(bd + c)(n-i)a_i - 2b(i+1)a_{i+1}] v^{2i} w^{n-i-1} = 0,$$

where $a_{n+1} = 0$. It implies that for $i = 0, 1, \dots, n$,

$$(bd + c)(n-i)a_i - 2b(i+1)a_{i+1} = 0, \quad (3.21)$$

where $a_{n+1} = 0$. We consider (3.21) in the two cases, (i) $bd = -c$, (ii) $bd \neq -c$.

- (i) For $bd = -c$, it follows from equation (3.21) that $2bia_i = 0, i = 1, 2, \dots, n$. If there exists some i such that $a_i \neq 0, i = 1, 2, \dots, n$, then $b = 0, c = 0$, and $k_0 = 0$ from (3.20), which contradicts with the cofactor is nonzero. Combining with Remark 2, we have $a_i = 0, i = 1, 2, \dots, n$, and $a_0 \neq 0$. In addition, $k_0 = \frac{4}{3}nc, c \neq 0$ from (3.20).
- (ii) For $bd \neq -c$, if $b = 0$ then $c \neq 0$ and $c(n-i)a_i = 0, i = 0, 1, \dots, n-1$ from (3.21). If there exists $i_0 \in \{0, 1, \dots, n-1\}$ such that $a_{i_0} \neq 0$, then $c = 0$. It contradicts with $c \neq 0$. Thus, $a_i = 0, i = 0, 1, \dots, n-1$, and $a_n \neq 0$. However, in this condition $k_0 = \frac{4}{3}(n-n)c = 0$, so there exists no Darboux polynomial with nonzero cofactor when $b = 0$. Therefore, $b \neq 0$.

If $b \neq 0$, it follows from (3.21) that $a_i \neq 0, i = 0, 1, \dots, n$. Besides, by the fact that the cofactor k_0 of one Darboux polynomial is unique, it is easy to get from (3.20) that

$$bd = -\frac{2}{3}c \text{ and } k_0 = \frac{4}{3}nc,$$

which implies that $c \neq 0$. In addition, it follows from $ibdma_i = 0$ in (3.18) that $m = 0$.

By the previous analysis, we know that if system (3.1) have Darboux polynomials f with nonzero cofactor k_0 , then there are two cases for F_0 corresponding to f

(i) $a_0 \neq 0, a_i = 0, i = 1, 2, \dots, n$ and $bd = -c, k_0 = \frac{4}{3}nc, c \neq 0$;

(ii) $a_i \neq 0, i = 0, 1, \dots, n$ and $bd = -\frac{2}{3}c, k_0 = \frac{4}{3}nc, c \neq 0, m = 0$.

In the following, we discuss these two cases, respectively.

First, for the case (i), let $m = 0$, then we have from (3.14) and (3.19)

$$F_0 = a_0 \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^n$$

and

$$F_1 = -\frac{1}{3}(a+1)na_0 \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-1} x^3 + na_0 \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-1} xy + \frac{1}{3}cna_0 \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-1} xz.$$

Then solving (3.5) with $j = 2$ for fixed v and w and substituting the transformation (3.7), we get

$$\begin{aligned} F_2 = & \left(-\frac{1}{9}nc^2 + \frac{1}{6}c^2 + \frac{1}{2}a \right) na_0 \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-1} x^2 + \frac{1}{2}n(n-1)a_0 y^2 \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-2} x^2 \\ & - \frac{1}{3}(a+1)n(n-1)a_0 y \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-2} x^4 \\ & + \left[\frac{1}{18}(a+1)^2 + \frac{1}{36}c^2 \right] n(n-1)a_0 \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-2} x^6 \\ & - \frac{1}{9}c(a+1)na_0 \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-1} z + \frac{1}{3}cn(n-1)a_0 y \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-2} x^2 z \\ & - \frac{1}{9}c(a+1)n(n-1)a_0 \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-2} x^4 z + G_2(x, y, z), \end{aligned}$$

where $G_2(x, y, z) = \overline{G}_2(v, w) = \sum_{i=1}^n \overline{a}_i y^{2i-1} \left(\frac{1}{4}x^4 - \frac{1}{2}z^2 \right)^{n-i}$, which is an arbitrary weight polynomial with degree $4n - 2$.

Next, solving (3.5) with $j = 3$ for fixed v and w , we get

$$\begin{aligned} \overline{F}_3 = & \left(\frac{1}{9}(a+1)na_0 + \frac{1}{3}\overline{a}_1 \right) cvw^{n-1} \int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\ & + \sum_{i=2}^n \left(-\frac{2}{3}i + 1 \right) c\overline{a}_i v^{2i-1} w^{n-i} \int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\ & + \left(\frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c - b \right) na_0 w^{n-1} \int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du + \overline{R}_3(u, v, w), \end{aligned}$$

where

$$\begin{aligned} \overline{R}_3(u, v, w) = & \left(\frac{2}{27}n - \frac{1}{9} \right) c^2 (a+1) na_0 w^{n-1} u - \frac{1}{3}(a+1)n(n-1)a_0 v^2 w^{n-2} u \\ & + \left(-\frac{1}{9}nc^2 + \frac{5}{18}c^2 + \frac{1}{2}a + \frac{1}{9}(a+1)^2 \right) n(n-1)a_0 v w^{n-2} u^3 \\ & + \frac{1}{6}n(n-1)(n-2)a_0 v^3 w^{n-3} u^3 \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{27}nc^2 - \frac{1}{9}c^2 - \frac{1}{6}a\right)(a+1)n(n-1)a_0w^{n-2}u^5 \\
& - \frac{1}{6}(a+1)n(n-1)(n-2)a_0v^2w^{n-3}u^5 \\
& + \left(\frac{1}{18}(a+1)^2 + \frac{1}{36}c^2\right)n(n-1)(n-2)a_0vw^{n-3}u^7 \\
& - \left(\frac{1}{162}(a+1)^2 + \frac{1}{108}c^2\right)(a+1)n(n-1)(n-2)a_0w^{n-3}u^9 \\
& - \frac{2}{9}c(a+1)n(n-1)a_0vw^{n-2}u\sqrt{\frac{1}{2}u^4 - 2w} \\
& + \left(-\frac{1}{81}nc^2 + \frac{1}{27}(a+1)^2 + \frac{7}{162}c^2 + \frac{1}{6}a\right)cn(n-1)a_0w^{n-2}u^3\sqrt{\frac{1}{2}u^4 - 2w} \\
& + \frac{1}{6}cn(n-1)(n-2)a_0v^2w^{n-3}u^3\sqrt{\frac{1}{2}u^4 - 2w} \\
& - \frac{1}{9}(a+1)cn(n-1)(n-2)a_0vw^{n-3}u^5\sqrt{\frac{1}{2}u^4 - 2w} \\
& + \left(\frac{1}{54}(a+1)^2 + \frac{1}{324}c^2\right)cn(n-1)(n-2)a_0w^{n-3}u^7\sqrt{\frac{1}{2}u^4 - 2w}.
\end{aligned}$$

The coefficients of $\int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ and $\int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ in \overline{F}_3 also should vanish. Thus,

$$\overline{a}_1 = -\frac{1}{3}(a+1)na_0, \quad \overline{a}_i = 0, i = 2, \dots, n \quad (3.22)$$

and

$$b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c. \quad (3.23)$$

Substituting the variable change (3.7) and the relationship (3.22), (3.23), we have $F_3(x, y, z) = \overline{R}_3(u, v, w)$.

Continuing to solve (3.5) with $j = 4$, we get

$$\overline{F}_4 = \left(-\frac{1}{81}c^3 - \frac{1}{27}a^2c + \frac{4}{27}ac - \frac{1}{27}c\right)(a+1)na_0w^{n-1} \int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du + \overline{R}_4(u, v, w),$$

where $R_4(x, y, z) = \overline{R}_4(u, v, w)$ is a weight polynomial of weight degree $4n - 4$. Then the coefficient of $\int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ should vanish. Hence we have

$$-\frac{1}{81}c^3 - \frac{1}{27}a^2c + \frac{4}{27}ac - \frac{1}{27}c = 0 \text{ or } a = -1.$$

According to the above analysis, for Case (i) $F_0 = a_0(\frac{1}{4}x^4 - \frac{1}{2}z^2)^n$, there are two different parameter conditions for system (1.2):

$$(1) \ a = -1, bd = -c, b = \frac{2}{27}c^3 - \frac{1}{3}c, c \neq 0.$$

$$(2) \ -\frac{1}{81}c^2 - \frac{1}{27}a^2 + \frac{4}{27}a - \frac{1}{27} = 0, bd = -c, b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c, c \neq 0.$$

Secondly, we analyze case (ii) for F_0 , it follows from (3.14) and (3.19) that

$$F_0 = \sum_{i=0}^n a_i y^{2i} \left(\frac{1}{4} x^4 - \frac{1}{2} z \right)^{n-i},$$

$$\begin{aligned} \overline{F}_1 = & - \sum_{i=0}^n \frac{1}{3} (a+1)(n-i) a_i v^{2i} w^{n-i-1} u^3 + \sum_{i=0}^n (n-i) a_i v^{2i+1} w^{n-i-1} u \\ & - \sum_{i=0}^n \frac{1}{3} c(n-i) a_i v^{2i} w^{n-i-1} u \sqrt{\frac{1}{2} u^4 - 2w}. \end{aligned}$$

Solving (3.5) with $j = 2$ for fixed v, w , we get

$$\overline{F}_2 = \sum_{i=0}^n \left[\frac{1}{3} c(n-i) a_i - 2b(i+1) a_{i+1} \right] v^{2i+1} w^{n-i-1} \int \frac{u}{\sqrt{\frac{1}{2} u^4 - 2w}} du + \overline{R}_2(u, v, w),$$

where \overline{R}_2 is a weight polynomial of weight degree $4n - 2$. Because the coefficients of $\int \frac{u}{\sqrt{\frac{1}{2} u^4 - 2w}} du$ must vanish, we have $i = 0, 1, \dots, n$

$$\frac{1}{3} c(n-i) a_i - 2b(i+1) a_{i+1} = 0.$$

Substituting the above equality, the form of \overline{F}_2 is as following

$$\begin{aligned} \overline{F}_2 = & \sum_{i=0}^n \left(-\frac{1}{9} (n-i) c^2 + \frac{1}{6} c^2 + \frac{a}{2} \right) (n-i) a_i v^{2i} w^{n-i-1} u^2 \\ & + \sum_{i=0}^n \frac{1}{2} (n-i)(n-i-1) a_i v^{2i+2} w^{n-i-2} u^2 \\ & - \sum_{i=0}^n \frac{1}{3} (a+1)(n-i)(n-i-1) a_i v^{2i+1} w^{n-i-2} u^4 \\ & + \sum_{i=0}^n \left[\frac{1}{18} (a+1)^2 + \frac{1}{36} c^2 \right] (n-i)(n-i-1) a_i v^{2i} w^{n-i-2} u^6 \\ & - \sum_{i=0}^n \frac{1}{9} c(a+1)(n-i) a_i v^{2i} w^{n-i-1} \sqrt{\frac{1}{2} u^4 - 2w} \\ & + \sum_{i=0}^n \frac{1}{3} c(n-i)(n-i-1) a_i v^{2i+1} w^{n-i-2} u^2 \sqrt{\frac{1}{2} u^4 - 2w} \\ & - \sum_{i=0}^n \frac{1}{9} c(a+1)(n-i)(n-i-1) a_i v^{2i} w^{n-i-2} u^4 \sqrt{\frac{1}{2} u^4 - 2w} \\ & + \sum_{i=1}^n \overline{a}_i v^{2i-1} w^{n-i}. \end{aligned}$$

Substituting changes of variables (3.7), $F_2(x, y, z) = \bar{F}_2(u, v, w)$.

Next, solving (3.5) with $j = 3$ for fixed v and w , we get

$$\begin{aligned}\bar{F}_3 &= \sum_{i=0}^n \left[\frac{1}{9}(a+1)(n-i)a_i + \frac{2}{3}\bar{a}_{i+1} \right] c v^{2i+1} w^{n-i-1} \int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\ &+ \sum_{i=0}^n \left(\frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c - b \right) (n-i)a_i v^{2i} w^{n-i-1} \int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du \\ &+ \bar{R}_3(u, v, w),\end{aligned}$$

where \bar{R}_3 is weight polynomial of weight degree $4n - 3$. Because the coefficients of $\int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ and $\int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du$ must vanish, we have $i = 0, 1, \dots, n$

$$\bar{a}_{i+1} = -\frac{1}{6}(a+1)(n-i)a_i, \quad b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c.$$

Substituting the above relationships of parameters in $\bar{R}_3(u, v, w)$, \bar{F}_3 becomes

$$\begin{aligned}\bar{F}_3 &= \sum_{i=0}^n \left(\frac{2}{27}(n-i) - \frac{1}{9} \right) c^2 (a+1)(n-i)a_i v^{2i} w^{n-i-1} u \\ &+ \sum_{i=0}^n (n-i)\bar{a}_i v^{2i} w^{n-i-1} u \\ &+ \sum_{i=0}^n \left(-\frac{1}{9}(n-i)c^2 + \frac{17}{54}c^2 + \frac{1}{2}a \right) (n-i)(n-i-1)a_i v^{2i+1} w^{n-i-2} u^3 \\ &+ \sum_{i=0}^n \frac{1}{6}(n-i)(n-i-1)(n-i-2)a_i v^{2i+3} w^{n-i-3} u^3 \\ &- \sum_{i=0}^n \frac{2}{9}b i c (n-i)a_i v^{2i-1} w^{n-i-1} u^3 \\ &- \sum_{i=0}^n \frac{1}{3}(a+1)(n-i)\bar{a}_i v^{2i-1} w^{n-i-1} u^3 \\ &+ \sum_{i=0}^n \left(\frac{1}{27}(n-i)c^2 - \frac{1}{9}c^2 - \frac{1}{6}a \right) (a+1)(n-i)(n-i-1)a_i v^{2i} w^{n-i-2} u^5 \\ &- \sum_{i=0}^n \frac{1}{6}(a+1)(n-i)(n-i-1)(n-i-2)a_i v^{2i+2} w^{n-i-3} u^5 \\ &+ \sum_{i=0}^n \left(\frac{1}{18}(a+1)^2 + \frac{1}{36}c^2 \right) (n-i)(n-i-1)(n-i-2)a_i v^{2i+1} w^{n-i-3} u^7 \\ &- \sum_{i=0}^n \left(\frac{1}{162}(a+1)^2 + \frac{1}{108}c^2 \right) (a+1)(n-i)(n-i-1)(n-i-2)a_i v^{2i} w^{n-i-3} u^9\end{aligned}$$

$$\begin{aligned}
& - \sum_{i=0}^n \frac{5}{27} c(a+1)(n-i)(n-i-1) a_i v^{2i+1} w^{n-i-2} u \sqrt{\frac{1}{2} u^4 - 2w} \\
& + \sum_{i=0}^n \frac{4}{9} b i(a+1)(n-i) a_i v^{2i-1} w^{n-i-1} u \sqrt{\frac{1}{2} u^4 - 2w} \\
& + \sum_{i=0}^n \frac{1}{3} c(n-i) \bar{a}_i v^{2i-1} w^{n-i-1} u \sqrt{\frac{1}{2} u^4 - 2w} \\
& + \sum_{i=0}^n \left(-\frac{1}{81} (n-i) c^2 + \frac{1}{27} (a+1)^2 + \frac{7}{162} c^2 + \frac{1}{6} a \right) \\
& c(n-i)(n-i-1) a_i v^{2i} w^{n-i-2} u^3 \sqrt{\frac{1}{2} u^4 - 2w} \\
& + \sum_{i=0}^n \frac{1}{6} c(n-i)(n-i-1)(n-i-2) a_i v^{2i+2} w^{n-i-3} u^3 \sqrt{\frac{1}{2} u^4 - 2w} \\
& - \sum_{i=0}^n \frac{1}{9} c(a+1)(n-i)(n-i-1)(n-i-2) a_i v^{2i+1} w^{n-i-3} u^5 \sqrt{\frac{1}{2} u^4 - 2w} \\
& + \sum_{i=0}^n \frac{1}{9} \left(\frac{1}{6} (a+1)^2 + \frac{1}{36} c^2 \right) c(n-i)(n-i-1)(n-i-2) a_i v^{2i} w^{n-i-3} u^7 \sqrt{\frac{1}{2} u^4 - 2w}.
\end{aligned}$$

Moreover, solving (3.5) with $j = 4$ for fixed v and w , we get

$$\bar{F}_4 = \sum_{i=0}^n \left(-\frac{1}{27} c^3 - \frac{1}{54} a^2 c + \frac{7}{54} a c - \frac{1}{54} c \right) (a+1)(n-i) a_i v^{2i} w^{n-i-1} \int \frac{u}{\sqrt{\frac{1}{2} u^4 - 2w}} du + \bar{R}_4$$

where \bar{R}_4 is weight polynomial of weight degree $4n-4$. The coefficient of $\int \frac{u}{\sqrt{\frac{1}{2} u^4 - 2w}} du$ must vanish. Then we have

$$-\frac{1}{27} c^3 - \frac{1}{54} a^2 c + \frac{7}{54} a c - \frac{1}{54} c = 0 \text{ or } a = -1.$$

According to the above analysis, for Case (ii) $F_0 = \sum_{i=0}^n a_i y^{2i} (\frac{1}{4} x^4 - \frac{1}{2} z)^{n-i}$, there are also two different conditions of parameters of system (1.2):

$$(3) \ a = -1, bd = -\frac{2}{3}c, b = \frac{2}{27}c^3 - \frac{1}{3}c, c \neq 0;$$

$$(4) \ -\frac{1}{81}c^2 - \frac{1}{27}a^2 + \frac{4}{27}a - \frac{1}{27} = 0, bd = -\frac{2}{3}c, b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c, c \neq 0.$$

Above all, the proof of Lemma (3.3) has been completed. \square

Remark 4. According to Lemma 3.3, the cofactor of system (1.2) must be $\frac{4}{3}nc$, where n is in the form of F_0 in (3.14).

Assume that $\phi(x, y, z)$ is an irreducible Darboux polynomial of system (1.2), then $\Phi(x, y, z) = \phi^n(x, y, z)$ can be expanded on the basis of characteristic curves (3.6),

which can be denoted as $\Phi(x, y, z) = \Phi_0(x, y, z) + \Phi_1(x, y, z) + \dots + \Phi_l(x, y, z)$, where $\Phi_j(x, y, z)$, $j = 0, 1, \dots, l$ are weight polynomials with weight exponents $(1, 2, 2)$, weight degree of $\Phi_j(x, y, z)$ is $l - j$. Due to this fact, we have the following result.

Lemma 3.4. *Assume that $\phi(x, y, z)$ is an irreducible Darboux polynomial of system (1.2) with cofactor \bar{k} . Let $\Phi(x, y, z) = \phi^n(x, y, z) = \Phi_0 + \Phi_1 + \dots + \Phi_l$. If $\Phi_0 = F_0$, where F_0 is in the form of (3.14), $\Phi(x, y, z)$ and $f(x, y, z)$ satisfy the same conditions of parameters and have the same cofactor, then $f(x, y, z)|_{m=0} = \Phi(x, y, z)$, where $f = F|_{\alpha=1} = F_0 + F_1 + \dots + F_l$.*

Proof. According to Remark 4, the cofactor of $f|_{m=0}$ is $\frac{4}{3}nc$. $\Phi(x, y, z)$ is a Darboux polynomial with cofactor $n\bar{k}$, then $\bar{k} = \frac{4}{3}c$.

Assume that $f|_{m=0}(x, y, z) \neq \Phi(x, y, z)$, then $\Phi - f|_{m=0}$ is a Darboux polynomial of system (1.2) with cofactor $\frac{4}{3}nc$. We also expand $\Phi - f|_{m=0}$ on the basis of (3.6), then we have $\Phi - f|_{m=0} = \eta_0 + \eta_1 + \dots + \eta_t$, where $\eta_j(x, y, z)$, $j = 0, 1, \dots, t$ are weight polynomials with weight exponents $(1, 2, 2)$, the weight degree of $\eta_j(x, y, z)$ is $t - j$.

If F_0 is in the form of (3.14), it follows from Remark 3 and $\Phi_0 = F_0$ that $l = 4n$ and $t = 4p, p \leq n - 1$. Thus, according to Remark 4, the cofactor of $\Phi - f|_{m=0}$ is $\frac{4}{3}pc$. Because the cofactor for one Darboux polynomial is unique, Then we have $f(x, y, z) = \Phi(x, y, z)$. \square

Lemma 3.5. *If F_0 is in the form (3.14), system (1.2) have irreducible Darboux polynomials with nonzero cofactor $\frac{4}{3}c$ and $c \neq 0$ shown in Table 1.*

Proof. According to Lemma 3.3, we have that the parameters in system (1.2) should satisfy conditions (1), (2), (3), (4) which are listed in the lemma.

If parameters of system (1.2) satisfy condition (1), ϕ_1 is a Darboux polynomial of system (1.2) with the cofactor $\frac{4}{3}c$ by a simple calculation, where

$$\phi_1(x, y, z) = \frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz + (\frac{1}{9}c^2 - 1)x^2.$$

Let $\Phi(x, y, z) = \phi_1^n(x, y, z)$, and expand $\Phi(x, y, z) = \Phi_0 + \Phi_1 + \dots + \Phi_{4n}$, where $\Phi_j(x, y, z)$ is a weight homogeneous polynomial of weight degree $4n - j$, weight exponents of Φ_j is $(1, 2, 2)$. It is easy to check $\Phi_0 = F_0$. Then, it follows from Lemma 3.4 that $f|_{m=0} = \Phi(x, y, z)$. It is to say, $F_0 + F_1 + \dots + F_{4n} = \phi_1^n$. Thus, $\phi_1(x, y, z)$ is an irreducible Darboux polynomial of system (1.2) with cofactor $\frac{4}{3}c$ under condition (1) of parameters.

If the parameters of system (1.2) satisfy condition (2), (3), (4), ϕ_2, ϕ_3 and ϕ_4 are Darboux polynomials of system (1.2) with the cofactor $\frac{4}{3}c$ by a simple calculation, where

$$\phi_2 = \frac{1}{2}x^4 - z^2 - \frac{2}{3}(a+1)x^3 + 2xy + \frac{2}{3}cxz - \frac{2}{9}c(a+1)z + (\frac{1}{9}c^2 + a)x^2 - \frac{2}{3}(a+1)y - \frac{2}{27}c^2(a+1)x$$

$$\phi_3 = \frac{1}{2}x^4 - z^2 - \frac{1}{2}dy^2 + 2xy + \frac{2}{3}cxz + (\frac{1}{9}c^2 - 1)x^2,$$

$$\begin{aligned}\phi_4 = & \frac{1}{2}x^4 - z^2 - \frac{1}{2}dy^2 - \frac{2}{3}(a+1)x^3 + 2xy + \frac{2}{3}cxz \\ & - \frac{2}{9}c(a+1)z + (\frac{1}{9}c^2 + a)x^2 - \frac{1}{3}(a+1)y - \frac{2}{27}c^2(a+1)x.\end{aligned}$$

It follows from Lemma 3.3 and 3.4 that ϕ_2, ϕ_3 and ϕ_4 are irreducible Darboux polynomials of system(1.2) with cofactor $\frac{4}{3}c$ under condition (2), (3), (4) of parameters. The proof of these conditions are the same as the proof of condition (1). \square

3.3 $k_0 = 0$

In this part, we classify Darboux polynomials with cofactor zero. In fact, Darboux polynomial with cofactor zero is the polynomial integral of the system. The method we used in this part is the same as the one which is used in the former section. As we know, no matter the cofactor k_0 is zero or not, F_0 should be in the form of (3.13) and (3.14). When $k_0 = 0$ and Darboux polynomials are not only depend on variable y , we have the following result.

Lemma 3.6. *If $k_0 = 0$, and Darboux polynomials of system (1.2) are not only depend on variable y , then the parameters b and c in system (1.2) are equal to zero.*

Proof. If F_0 is in the form (3.13), the equality (3.16) in the proof of Lemma 3.2 holds whether $k_0 = 0$ or not. It follows from (3.16) that $ba_1 = 0$.

If $b \neq 0$, then $a_1 = 0$, it is from (3.16) that $a_i = 0, i = 1, \dots, n$, then $F_0 = 0$, which contradicts with Remark 2. So $b = 0$.

If $b = 0$, (3.16) becomes

$$\sum_{i=1}^{n-1} c(n-i)a_i v^{2i} w^{n-i-1} = 0.$$

If there exists $a_i \neq 0, i = 1, 2, \dots, n-1$, then $c = 0$; If $a_n \neq 0, a_i = 0, i = 1, 2, \dots, n-1$, then $F_0 = a_n y^{2n-1}$, and the polynomial integral is only depend on variable y . Thus, if $b = 0$, then $c = 0$.

If F_0 is in the form (3.14), the equalities (3.20) and (3.21) in the proof of Lemma 3.3 hold whether $k_0 = 0$ or not. We discuss this situation also in the following cases:

(i) $bd = -c$

The equality $\frac{2}{3}(2n+i)c = 0$ holds from (3.20), then $c = 0$. Substituting $bd = -c$ into (3.21), we have $2bia_i = 0, i = 1, 2, \dots, n$. If there exists $a_i \neq 0, i = 1, 2, \dots, n$, then $b = 0$. If $a_0 \neq 0, a_i = 0, i = 1, 2, \dots, n$, then $F_0 = a_0(\frac{1}{4}x^4 - \frac{1}{2}z^2)^n$, and it follows from (3.23) that $b = 0$.

(ii) $bd \neq -c$

If $b = 0$, then $c \neq 0$, it follows from (3.21) that $c(n-i)a_i = 0, i = 0, 1, \dots, n$, then $a_i = 0, i = 0, 1, \dots, n-1$. It follows from Remark 2 that $a_n \neq 0, a_i = 0, i =$

$0, 1, \dots, n-1$. Let $m = 0$, then $F_0 = a_n y^{2n}$ and $f|_{m=0} = y^{2n}$. It contradicts with f is not only depend on y .

If $b \neq 0, c = 0$, then $bd \neq 0$. From (3.20) we have $-2ibd = 0$, then $a_0 \neq 0, a_i = 0, i = 1, 2, \dots, n$. and $F_0 = a_0(\frac{1}{4}x^4 - \frac{1}{2}z^2)^n$. Substituting F_0 and $c = 0, k_0 = 0$ in (3.5), when $j = 2$ we have $bd = 0$. which contradicts with $bd \neq 0$.

If $b \neq 0, c \neq 0$, it follows from (3.21) that $a_i \neq 0, i = 0, 1, \dots, n$. Because the cofactor of a Darboux polynomial is unique, we have $k_0 = \frac{4}{3}nc = 0, bd = -\frac{2}{3}c$, which contradicts with $c \neq 0$.

So, the relation $bd \neq -c$ doesn't hold.

The lemma has been proved. \square

In the following, we just need to consider the situation when $b = c = 0$. And it is obvious that y is an irreducible Darboux polynomial of system (1.2) when $k_0 = 0$.

If $k_0 = 0, b = c = 0$, (3.5) becomes

$$\begin{aligned} L[F_1] &= [(a+1)x^2 - y] \frac{\partial F_0}{\partial z}, \\ L[F_j] &= [(a+1)x^2 - y] \frac{\partial F_{j-1}}{\partial z} - ax \frac{\partial F_{j-2}}{\partial z}, \quad j = 2, 3, \dots, l+3, \end{aligned} \quad (3.24)$$

where $F_j = 0, j > l$.

If $b = c = 0$, we consider $F_0^* = a_0(\frac{1}{4}x^4 - \frac{1}{2}z^2)^n$. Let $m = 0$, substituting F_0 into equations of (3.24) and solving the the differential equations, we can obtain the forms of $F_j^*, j = 1, 2, \dots, 4n$. The calculation procedure is the same as the one which is used in the proof when $k_0 \neq 0$.

If $m = 0, b = c = 0$,

$$\phi_5 = \frac{1}{4}x^4 - \frac{1}{2}z^2 - \frac{1}{3}(a+1)x^3 + xy + \frac{1}{2}ax^2$$

is a polynomial integral of system (1.2).

It follows from Lemma 3.3 and 3.4 that $\phi_5^n(x, y, z) = F_0^* + F_1^* + \dots + F_{4n}^*$. The proof is the same as that in the proof of Lemma 4.

Thus, $\frac{1}{4}x^4 - \frac{1}{2}z^2 - \frac{1}{3}(a+1)x^3 + xy + \frac{1}{2}ax^2$ and y are polynomial integrals of system (1.2).

When F_0 is in the general form of (3.13) and (3.14), by calculating equations in (3.24), we obtain a polynomial integral of system (1.2) denoted by $F(x, y, z)$. The relationship between $F(x, y, z)$ and y, ϕ_5 is shown in the following Lemma.

Lemma 3.7. *If $m = 0, F_0 = y^p(\frac{1}{4}x^4 - \frac{1}{2}z^2)^q, p, q \in \mathbb{N}$, then the polynomial integral calculated by (3.24) with F_0 is $y^p \phi_5^q$.*

Proof. Obviously, if $\tilde{F}_0 = (\frac{1}{4}x^4 - \frac{1}{2}z^2)^q$, then the Darboux polynomial calculated by \tilde{F}_0 is ϕ_5^q .

Assume that when $\tilde{F}_0 = (\frac{1}{4}x^4 - \frac{1}{2}z^2)^q$, solving equations in (3.24) we have $\tilde{F}_j, j = 1, 2, \dots, 4q$.

Notice that if $m = 0$, differential equations in (3.24) are linear and don't have relationship with the partial derivatives of variable y . Thus, when $F_0 = y^p(\frac{1}{4}x^4 - \frac{1}{2}z^2)^q$, we have $F_j = y^p \tilde{F}_j, j = 1, 2, \dots, 4q$.

Thus, the polynomial integral calculated by $F_0 = y^p(\frac{1}{4}x^4 - \frac{1}{2}z^2)^q$ is $y^p \phi_5^q$. \square

According to Lemma 3.7 and the linearity of equations in (3.24), we have the conclusion that if $m = 0$, and F_0 is in the form of (3.13) or (3.14), the polynomial integral can be generated by y and ϕ_5 .

[The proof of Theorem 1] According to Lemma 3.5 and the conclusion we obtained when $k_0 = 0$, we have a complete classification of invariant algebraic surfaces of F-N system (1.2).

3.4 Proof of the corollary

Proof. (a) It is obvious from Theorem 1.

(b) It follows from theorem 1 that system (1.2) has two functional independent polynomial integrals, then F-N system (1.2) is polynomial integrable. \square

Appendices

The following formulas are used in the proof of Theorem 1:

Denoting $E = \text{EllipticE}(\frac{1}{2}u\sqrt{-\frac{2}{\sqrt{w}}}, I)$, $F = \text{EllipticF}(\frac{1}{2}u\sqrt{-\frac{2}{\sqrt{w}}}, I)$.

$$\begin{aligned} \int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du &= \frac{\sqrt{\frac{4+2u^2}{\sqrt{w}}} \sqrt{\frac{\sqrt{4-2u^2}}{\sqrt{w}}} F}{\sqrt{-\frac{2}{\sqrt{w}}} \sqrt{2u^4 - 8w}}; \\ \int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du &= 2\sqrt{w} \frac{\sqrt{\frac{4+2u^2}{\sqrt{w}}} \sqrt{\frac{\sqrt{4-2u^2}}{\sqrt{w}}} (F - E)}{\sqrt{-\frac{2}{\sqrt{w}}} \sqrt{2u^4 - 8w}}; \\ \int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du &= \frac{\sqrt{2}}{2} \ln(\sqrt{2}u^2 + \sqrt{2u^4 - 8w}). \end{aligned}$$

Next, we denote $A = \int \frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}} du$, $B = \int \frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}} du$, $C = \int \frac{u}{\sqrt{\frac{1}{2}u^4 - 2w}} du$, then other integrals in the calculation can be expressed by A , B and C . The formulas are presented as following:

$$\int \sqrt{\frac{1}{2}u^4 - 2w} du = \frac{1}{3}u\sqrt{\frac{1}{2}u^4 - 2w} - \frac{4}{3}wA;$$

$$\begin{aligned}
\int u\sqrt{\frac{1}{2}u^4 - 2w}du &= \frac{1}{4}u^2\sqrt{\frac{1}{2}u^4 - 2w} - wC; \\
\int \frac{u^4}{\sqrt{\frac{1}{2}u^4 - 2w}}du &= \frac{2}{3}u\sqrt{\frac{1}{2}u^4 - 2w} + \frac{4}{3}wA; \\
\int \frac{u^5}{\sqrt{\frac{1}{2}u^4 - 2w}}du &= \frac{1}{2}u^2\sqrt{\frac{1}{2}u^4 - 2w} + 2wC; \\
\int \frac{u^6}{\sqrt{\frac{1}{2}u^4 - 2w}}du &= \frac{2}{5}u^3\sqrt{\frac{1}{2}u^4 - 2w} + \frac{12}{5}wB; \\
\int \frac{u^9}{\sqrt{\frac{1}{2}u^4 - 2w}}du &= (\frac{1}{4}u^6 + \frac{3}{2}w^2u)\sqrt{\frac{1}{2}u^4 - 2w} + 6w^2C; \\
\int u^2\sqrt{\frac{1}{2}u^4 - 2w}du &= \frac{1}{5}u^3\sqrt{\frac{1}{2}u^4 - 2w} - \frac{4}{5}wB; \\
\int u^4\sqrt{\frac{1}{2}u^4 - 2w}du &= (\frac{1}{7}u^5 - \frac{8}{21}wu)\sqrt{\frac{1}{2}u^4 - 2w} - \frac{16}{21}w^2A; \\
\int u^5\sqrt{\frac{1}{2}u^4 - 2w}du &= (\frac{1}{8}u^6 - \frac{1}{4}wu^2)\sqrt{\frac{1}{2}u^4 - 2w} - w^2C; \\
\int u^6\sqrt{\frac{1}{2}u^4 - 2w}du &= (\frac{1}{9}u^7 - \frac{8}{45}wu^3)\sqrt{\frac{1}{2}u^4 - 2w} - \frac{16}{15}w^2B; \\
\int u^9\sqrt{\frac{1}{2}u^4 - 2w}du &= (\frac{1}{12}u^{10} - \frac{1}{12}wu^6 - \frac{1}{2}w^2u^2)\sqrt{\frac{1}{2}u^4 - 2w} - 2w^3A; \\
\int (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{-1}du &= \frac{\sqrt{2}}{24}u^3w^{-1} - \frac{1}{12}uw^{-1}\sqrt{\frac{1}{2}u^4 - 2w} - \frac{8}{3}A; \\
\int (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{-1}\frac{1}{\sqrt{\frac{1}{2}u^4 - 2w}}du &= -\frac{1}{4}uw^{-1} - \frac{\sqrt{2}}{8}w^{-1}B; \\
\int (\sqrt{2}u^2 + 2\sqrt{\frac{1}{2}u^4 - 2w})^{-1}\frac{u^2}{\sqrt{\frac{1}{2}u^4 - 2w}}du &= -\frac{1}{12}u^3w^{-1} + \frac{\sqrt{2}}{12}uw^{-1}\sqrt{\frac{1}{2}u^4 - 2w} + \frac{\sqrt{2}}{6}A.
\end{aligned}$$

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